

Systems of four coupled one sided Sylvester-type real quaternion matrix equations and their applications¹

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Abstract: In this paper, we derive some necessary and sufficient solvability conditions for some systems of one sided coupled Sylvester-type real quaternion matrix equations in terms of ranks and generalized inverses of matrices. We also give the expressions of the general solutions to these systems when they are solvable. Moreover, we provide some numerical examples to illustrate our results. The findings of this paper extend some known results in the literature.

Keywords: Quaternion; Sylvester-type equations; Moore-Penrose inverse; Rank; Solution; Solvability

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1. Introduction

Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton in 1843. It is well known that quaternion algebra is an associative and noncommutative division algebra over the real number field. Quaternions and quaternion matrices have found a huge amount of applications in quantum physics, signal and color image processing, and so on (e.g. [3], [21], [27]-[30]). General properties of quaternions and quaternion matrices can be found in [48]. Quaternion matrix equations play an important role in mathematics and other disciplines, such as engineering, system and control theory. There have been many papers using various approaches to investigate many quaternion matrix equations (e.g. [9]-[11], [38]-[43], [45], [46], [49]).

The Sylvester-type matrix equations have wide applications in neural network [47], robust control ([4], [31]), output feedback control ([25], [26]), the almost noninteracting control by measurement feedback problem ([32], [44]), graph theory [7], and so on. Since Roth [23] first studied the one-sided generalized Sylvester matrix equation

$$AX - YB = C$$

over the complex field in 1952, there have been many papers to discuss the generalized Sylvester matrix equations (e.g. [1], [2], [8], [13], [17], [18], [20], [24], [34]-[37], [44]). For instance, De Terán et al. ([5], [6]) considered the \star -Sylvester equation $AX + X^*B = 0$ and $AX + BX^* = 0$. Quite recently, Dmytryshyn and Kågström [7] presented some solvability conditions of the following

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systems consisting of Sylvester and \star -Sylvester equations through the corresponding equivalence relations of the block matrices

$$\begin{cases} A_i X_k \pm X_j B_i = C_i, & i = 1, \dots, n_1, \\ F_{i'} X_{k'} \pm X_{j'}^* G_{i'} = H_{i'}, & i' = 1, \dots, n_2, \end{cases}$$

where $k, j, k', j' \in \{1, \dots, m\}$, each unknown X_l is $r_l \times c_l$, $l = 1, \dots, m$, and all other matrices are of appropriate sizes. Jonsson and Kågström ([15], [16]) provided some effective approaches for solving one-sided and two-sided triangular Sylvester-type matrix equations.

The study on the coupled generalized Sylvester matrix equations is active in recent years. Lee and Vu [19] derived a consistency condition for the following system of mixed Sylvester matrix equations through the corresponding equivalence relations of the block matrices

$$A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_3 - X_2 B_2 = C_2, \quad (1.1)$$

where A_i, B_i and C_i ($i = 1, 2$) are given matrices over a field, X_1, X_2 and X_3 are unknowns. Wang and He [34] gave some computable necessary and sufficient solvability conditions for the system (1.1), and presented the general solution when (1.1) is solvable. Afterwards, He and Wang [14] provided some necessary and sufficient solvability conditions for the system of mixed Sylvester matrix equations

$$A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_2 - X_3 B_2 = C_2, \quad (1.2)$$

where A_i, B_i and C_i ($i = 1, 2$) are given complex matrices, X_1, X_2 and X_3 are unknowns. They also derived an expression of the general solution to the system (1.2). Recently, Wang and He [35] considered the following three systems of generalized coupled Sylvester matrix equations with four variables

$$\begin{cases} A_1 X - Y B_1 = C_1, \\ A_2 Z - Y B_2 = C_2, \\ A_3 Z - W B_3 = C_3, \end{cases} \quad (1.3)$$

$$\begin{cases} A_1 X - Y B_1 = C_1, \\ A_2 Y - Z B_2 = C_2, \\ A_3 Z - W B_3 = C_3, \end{cases} \quad (1.4)$$

$$\begin{cases} A_1 X - Y B_1 = C_1, \\ A_2 Y - Z B_2 = C_2, \\ A_3 W - Z B_3 = C_3, \end{cases} \quad (1.5)$$

where A_i, B_i and C_i ($i = 1, 2, 3$) are given complex matrices, X, Y, Z and W are unknowns. He, Mauricio, Wang and De Moor [8] considered two sided coupled generalized Sylvester matrix equations with four variables

$$A_i X_i B_i + C_i X_{i+1} D_i = E_i, \quad i = 1, 2, 3,$$

where A_i, B_i, C_i, D_i, E_i , ($i = 1, 2, 3$) are given complex matrices, X_i are unknowns. Very recently, He and Wang [13] derived the solvability conditions and the general solution to the system of

the periodic discrete-time coupled Sylvester quaternion matrix equations

$$\begin{cases} A_k X_k + Y_k B_k = M_k, \\ C_k X_{k+1} + Y_k D_k = N_k, \end{cases} \quad (k = 1, 2),$$

where $A_k, B_k, C_k, D_k, M_k, N_k$ are given matrices, X_k and Y_k are unknowns.

To our best knowledge, there has been little information on the solvability and the general solutions to the systems of four coupled one sided Sylvester-type real quaternion matrix equations with five unknowns. Motivated by the wide applications of generalized Sylvester matrix equations and real quaternion matrix equations and in order to improve the theoretical development of generalized Sylvester real quaternion matrix equations, we in this paper consider the solvability and the expressions of the general solutions to the following systems of four coupled one sided Sylvester-type real quaternion matrix equations

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_3 - X_2 B_2 = C_2, \\ A_3 X_3 - X_4 B_3 = C_3, \\ A_4 X_4 - X_5 B_4 = C_4, \end{cases} \quad (1.6)$$

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \\ A_3 X_3 - X_4 B_3 = C_3, \\ A_4 X_5 - X_4 B_4 = C_4, \end{cases} \quad (1.7)$$

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \\ A_3 X_4 - X_3 B_3 = C_3, \\ A_4 X_5 - X_4 B_4 = C_4, \end{cases} \quad (1.8)$$

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \\ A_3 X_3 - X_4 B_3 = C_3, \\ A_4 X_4 - X_5 B_4 = C_4, \end{cases} \quad (1.9)$$

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_3 - X_2 B_2 = C_2, \\ A_3 X_4 - X_3 B_3 = C_3, \\ A_4 X_4 - X_5 B_4 = C_4, \end{cases} \quad (1.10)$$

where $A_i, B_i, C_i, (i = 1, 2, 3, 4)$ are given real quaternion matrices, and X_1, \dots, X_5 are unknowns. Note that the i th equation and $(i + 1)$ th equation in (1.6)-(1.10) have a common variable X_{i+1} . The given real quaternion matrices A_i located at the left of the variables and B_i located at the right of the variables. Systems (1.1)-(1.5) are special cases of systems (1.6)-(1.10).

The remainder of the paper is organized as follows. In Section 2, we provide some known lemmas which are used in this paper. In Section 3,4,5,6,7, we present some solvability conditions

to the systems of four coupled one sided Sylvester-type real quaternion matrix equations (1.6)-(1.10), respectively. We also derive the general solutions to the systems (1.6)-(1.10), respectively. Moreover, we give some numerical examples to illustrate our results.

Throughout this paper, let \mathbb{R} be the real number fields. Let $\mathbb{H}^{m \times n}$ be the set of all $m \times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For $A \in \mathbb{H}^{m \times n}$, the symbols A^* and $r(A)$ denote the conjugate transpose and the rank of A , respectively. The identity matrix with appropriate size is denoted by I . The Moore-Penrose inverse of $A \in \mathbb{H}^{m \times n}$, denoted by A^\dagger , is defined to be the unique solution X to the following four matrix equations

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) (AX)^* = AX, \quad (4) (XA)^* = XA.$$

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by A , respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$.

2. Preliminaries

In this section, we review some lemmas which are used in the further development of this paper. The following lemma give the solvability conditions and general solution to the mixed Sylvester real quaternion matrix equations (1.1).

Lemma 2.1. [34] *Let A_i, B_i , and $C_i (i = 1, 2)$ be given. Set*

$$D_1 = R_{B_1}B_2, A = R_{A_2}A_1, B = B_2L_{D_1}, C = R_{A_2}(R_{A_1}C_1B_1^\dagger B_2 - C_2)L_{D_1}.$$

Then the following statements are equivalent:

- (1) *The mixed Sylvester real quaternion matrix equations (1.1) is consistent.*
- (2) $R_{A_1}C_1L_{B_1} = 0, R_A C = 0, CL_B = 0.$
- (3)

$$r \begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1),$$

$$r \begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r \begin{pmatrix} B_2 & B_1 & 0 & 0 \\ C_2 & C_1 & A_1 & A_2 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2).$$

In this case, the general solution to the mixed Sylvester real quaternion matrix equations (1.6) can be expressed as

$$X = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} W_1,$$

$$Y = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1},$$

$$Z = A_2^\dagger (C_2 - R_{A_1} C_1 B_1^\dagger B_2 + A_1 U_1 B_2) + W_4 D_1 + L_{A_2} W_6,$$

where

$$U_1 = A^\dagger C B^\dagger + L_A W_2 + W_3 R_B,$$

$$V_1 = -R_{A_2}(C_2 - R_{A_1} C_1 B_1^\dagger B_2 + A_1 U_1 B_2) D_1^\dagger + A_2 W_4 + W_5 R_{D_1},$$

and W_1, \dots, W_6 are arbitrary matrices over \mathbb{H} with appropriate sizes.

The solvability conditions and general solution to the mixed Sylvester real quaternion matrix equations (1.2) can be found in the following lemma.

Lemma 2.2. [14] *Let A_i, B_i , and $C_i (i = 1, 2)$ be given. Set*

$$A_{11} = R_{(A_2 A_1)} A_2, \quad B_{11} = R_{B_1} L_{B_2}, \quad C_{11} = R_{(A_2 A_1)} (A_2 R_{A_1} C_1 B_1^\dagger + C_2) L_{B_2}.$$

Then the following statements are equivalent:

- (1) *The mixed generalized Sylvester real quaternion matrix equations (1.2) is consistent.*
- (2)

$$R_{A_1} C_1 L_{B_1} = 0, \quad R_{A_{11}} C_{11} = 0, \quad C_{11} L_{B_{11}} = 0.$$

(3)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad r \begin{pmatrix} A_2 A_1 & A_2 C_1 + C_2 B_1 \\ 0 & B_2 B_1 \end{pmatrix} = r(A_2 A_1) + r(B_2 B_1).$$

In this case, the general solution to the mixed generalized Sylvester real quaternion matrix equations (1.6) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} W_1,$$

$$X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1},$$

$$X_3 = -R_{(A_2 A_1)} (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) B_2^\dagger + A_2 A_1 W_4 + W_5 R_{B_2},$$

where

$$V_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_2 + W_3 R_{B_{11}},$$

$$U_1 = (A_2 A_1)^\dagger (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_1)} W_6,$$

and W_1, \dots, W_6 are arbitrary matrices over \mathbb{H} with appropriate sizes.

Based on Lemma 2.1, we can solve the following mixed Sylvester real quaternion matrix equations

$$A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_1 - X_3 B_2 = C_2. \quad (2.1)$$

Lemma 2.3. *Let A_i, B_i , and $C_i (i = 1, 2)$ be given. Set*

$$A_{11} = R_{(A_2 L_{A_1})} A_2, \quad B_{11} = B_1 L_{B_2}, \quad C_{11} = R_{(A_2 L_{A_1})} (C_2 - A_2 A_1^\dagger C_1) L_{B_2},$$

Then the following statements are equivalent:

- (1) *The mixed Sylvester real quaternion matrix equations (2.1) is consistent.*
- (2) $R_{A_1} C_1 L_{B_1} = 0, \quad R_{A_{11}} C_{11} = 0, \quad C_{11} L_{B_{11}} = 0.$
- (3)

$$r \begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1),$$

$$r \begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r \begin{pmatrix} C_1 & A_1 \\ C_2 & A_2 \\ B_1 & 0 \\ B_2 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + r \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

In this case, the general solution to the mixed Sylvester real quaternion matrix equations (2.1) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} U_2,$$

$$X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + W_6 R_{B_1},$$

$$X_3 = -R_{(A_2 L_{A_1})} (C_2 - A_2 A_1^\dagger C_1 - A_2 U_1 B_1) B_2^\dagger + A_2 L_{A_1} W_1 + W_3 R_{B_2},$$

where

$$U_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_4 + W_5 R_{B_{11}},$$

$$U_2 = (A_2 L_{A_1})^\dagger (C_2 - A_2 A_1^\dagger C_1 - A_2 U_1 B_1) + W_1 B_2 + L_{(A_2 L_{A_1})} W_2,$$

and W_1, \dots, W_6 are arbitrary matrices over \mathbb{H} with appropriate sizes.

The following real quaternion matrix equation

$$A_1 X_1 + X_2 B_1 + C_3 X_3 D_3 + C_4 X_4 D_4 = E_1 \tag{2.2}$$

which play an important role in the construction of the solvability conditions and the general solution to the systems (1.6)-(1.10).

Lemma 2.4. [12], [33] *Let $A_1, B_1, C_3, D_3, C_4, D_4$, and E_1 be given. Set*

$$A = R_{A_1} C_3, \quad B = D_3 L_{B_1}, \quad C = R_{A_1} C_4, \quad D = D_4 L_{B_1},$$

$$E = R_{A_1} E_1 L_{B_1}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M.$$

Then the equation (2.2) is consistent if and only if

$$R_M R_A E = 0, \quad E L_B L_N = 0, \quad R_A E L_D = 0, \quad R_C E L_B = 0.$$

In this case, the general solution can be expressed as

$$\begin{aligned}
X_1 &= A_1^\dagger(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^\dagger T_7 B_1 + L_{A_1} T_6, \\
X_2 &= R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^\dagger + A_1 A_1^\dagger T_7 + T_8 R_{B_1}, \\
X_3 &= A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S T_2 R_N D B^\dagger + L_A T_4 + T_5 R_B, \\
X_4 &= M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D,
\end{aligned}$$

where T_1, \dots, T_8 are arbitrary matrices over \mathbb{H} with appropriate sizes.

The following lemma can be easily generalized to \mathbb{H} .

Lemma 2.5. [22] Let $A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, C \in \mathbb{H}^{l \times n}, D \in \mathbb{H}^{m \times p}, E \in \mathbb{H}^{q \times n}, Q \in \mathbb{H}^{m_1 \times k}$, and $P \in \mathbb{H}^{l \times n_1}$ be given. Then

- (1) $r(A) + r(R_A B) = r(B) + r(R_B A) = r(A, B)$.
- (2) $r(A) + r(C L_A) = r(C) + r(A L_C) = r \begin{pmatrix} A \\ C \end{pmatrix}$.

3. Some solvability conditions and the general solution to system (1.6)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6). For simplicity, put

$$A_{11} = R_{B_2} B_1, B_{11} = R_{A_1} A_2, C_{11} = B_1 L_{A_{11}}, D_{11} = R_{A_1}(R_{A_2} C_2 B_2^\dagger B_1 - C_1) L_{A_{11}}, \quad (3.1)$$

$$A_{22} = R_{(A_4 A_3)} A_4, B_{22} = R_{B_3} L_{B_4}, C_{22} = R_{(A_4 A_3)}(A_4 R_{A_3} C_3 B_3^\dagger + C_4) L_{B_4}, \quad (3.2)$$

$$A_{33} = (L_{A_2}, -L_{A_3}), B_{33} = \begin{pmatrix} R_{C_{11}} B_2 \\ -B_4 B_3 \end{pmatrix}, A_{44} = -L_{(A_4 A_3)}, \quad (3.3)$$

$$E_1 = A_3^\dagger C_3 + (A_4 A_3)^\dagger (C_4 B_3 + A_4 R_{A_3} C_3) - A_2^\dagger C_2 - B_{11}^\dagger D_{11} C_{11}^\dagger B_2, \quad (3.4)$$

$$A = R_{A_{33}} L_{B_{11}}, B = B_2 L_{B_{33}}, C = R_{A_{33}} A_{44}, D = B_3 L_{B_{33}}, \quad (3.5)$$

$$E = R_{A_{33}} E_1 L_{B_{33}}, M = R_A C, N = D L_B, S = C L_M. \quad (3.6)$$

Now we give the fundamental theorem of this section.

Theorem 3.1. Let A_i, B_i , and $C_i (i = 1, 2, 3, 4)$ be given. Then the following statements are equivalent:

- (1) The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) is consistent.
- (2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \quad (3.7)$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2), \quad (3.8)$$

$$r \begin{pmatrix} A_4C_3 + C_4B_3 & A_4A_3 \\ B_4B_3 & 0 \end{pmatrix} = r(A_4A_3) + r(B_4B_3), \quad (3.9)$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & A_4C_3 + C_4B_3 & 0 & A_4A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_4B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_4B_3 \end{pmatrix}, \quad (3.10)$$

$$r \begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}, \quad (3.11)$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ C_3 & 0 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad (3.12)$$

$$r \begin{pmatrix} C_2 & A_2 \\ A_4C_3 + C_4B_3 & A_4A_3 \\ B_2 & 0 \\ B_4B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_4B_3 \end{pmatrix}. \quad (3.13)$$

(3)

$$R_{A_2}C_2L_{B_2} = 0, \quad D_{11}L_{C_{11}} = 0, \quad R_{B_{11}}D_{11} = 0, \quad (3.14)$$

$$R_{A_3}C_3L_{B_3} = 0, \quad R_{A_{22}}C_{22} = 0, \quad C_{22}L_{B_{22}} = 0, \quad (3.15)$$

$$R_M R_A E = 0, \quad E L_B L_N = 0, \quad R_A E L_D = 0, \quad R_C E L_B = 0. \quad (3.16)$$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) can be expressed as

$$X_1 = A_1^\dagger (C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6, \quad (3.17)$$

$$X_2 = -R_{A_2}C_2B_2^\dagger + A_2U_1 + V_1R_{B_2}, \quad X_4 = -R_{A_3}C_3B_3^\dagger + A_3U_2 + V_2R_{B_3}, \quad (3.18)$$

$$X_5 = -R_{(A_4A_3)}(C_4 + A_4R_{A_3}C_3B_3^\dagger - A_4V_2R_{B_3})B_4^\dagger + A_4A_3T_4 + T_5R_{B_4}, \quad (3.19)$$

$$X_3 = A_2^\dagger C_2 + U_1B_2 + L_{A_2}W_1, \quad \text{or } X_3 = A_3^\dagger C_3 + U_2B_3 + L_{A_3}T_1, \quad (3.20)$$

where

$$U_1 = B_{11}^\dagger D_{11}C_{11}^\dagger + L_{B_{11}}W_2 + W_3R_{C_{11}}, \quad (3.21)$$

$$V_1 = -R_{A_1}(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1)A_{11}^\dagger + A_1W_4 + W_5R_{A_{11}}, \quad (3.22)$$

$$V_2 = A_{22}^\dagger C_{22}B_{22}^\dagger + L_{A_{22}}T_2 + T_3R_{B_{22}}, \quad (3.23)$$

$$U_2 = (A_4A_3)^\dagger(C_4 + A_4R_{A_3}C_3B_3^\dagger - A_4V_2R_{B_3}) + T_4B_4 + L_{(A_4A_3)}T_6, \quad (3.24)$$

$$W_1 = (I_{p_1}, 0)[A_{33}^\dagger(E_1 - L_{B_{11}}W_2B_2 - A_{44}T_6B_3) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6], \quad (3.25)$$

$$T_1 = (0, I_{p_2})[A_{33}^\dagger(E_1 - L_{B_{11}}W_2B_2 - A_{44}T_6B_3) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6], \quad (3.26)$$

$$W_3 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 - A_{44}T_6B_3)B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix}, \quad (3.27)$$

$$T_4 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 - A_{44}T_6B_3)B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix}, \quad (3.28)$$

$$W_2 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1R_NDB^\dagger + L_AZ_2 + Z_3R_B, \quad (3.29)$$

$$T_6 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D, \quad (3.30)$$

the remaining W_j, T_j, Z_j are arbitrary matrices over \mathbb{H} , p_1 and p_2 are the column numbers of A_2 and A_3 , respectively, p_3 and p_4 are the row numbers of B_1 and B_4 , respectively.

Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) into two parts

$$\begin{cases} A_2X_3 - X_2B_2 = C_2, \\ A_1X_1 - X_2B_1 = C_1, \end{cases} \quad (3.31)$$

and

$$\begin{cases} A_3X_3 - X_4B_3 = C_3, \\ A_4X_4 - X_5B_4 = C_4. \end{cases} \quad (3.32)$$

Observe that system (3.31) has the form of (1.1), and system (3.32) has the form of (1.2). We can solve the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) through the following three steps. In the first step, we consider the system (3.31). It follows from Lemma 2.1 that the system (3.31) is consistent if and only if

$$r \begin{pmatrix} C_n & A_n \\ B_n & 0 \end{pmatrix} = r(A_n) + r(B_n), \quad (n = 1, 2), \quad (3.33)$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2), \quad (3.34)$$

or

$$R_{A_2}C_2L_{B_2} = 0, \quad R_{B_{11}}D_{11} = 0, \quad D_{11}L_{C_{11}} = 0. \quad (3.35)$$

In this case, the general solution to the system (3.31) can be expressed as

$$X_3 = A_2^\dagger C_2 + U_1 B_2 + L_{A_2} W_1, \quad (3.36)$$

$$X_2 = -R_{A_2} C_2 B_2^\dagger + A_2 U_1 + V_1 R_{B_2}, \quad (3.37)$$

$$X_1 = A_1^\dagger (C_1 - R_{A_2} C_2 B_2^\dagger B_1 + A_2 U_1 B_1) + W_4 A_{11} + L_{A_1} W_6, \quad (3.38)$$

where

$$U_1 = B_{11}^\dagger D_{11} C_{11}^\dagger + L_{B_{11}} W_2 + W_3 R_{C_{11}}, \quad (3.39)$$

$$V_1 = -R_{A_1} (C_1 - R_{A_2} C_2 B_2^\dagger B_1 + A_2 U_1 B_1) A_{11}^\dagger + A_1 W_4 + W_5 R_{A_{11}}, \quad (3.40)$$

and W_1, \dots, W_6 are arbitrary matrices over \mathbb{H} with appropriate sizes.

In the second step, we consider the system (3.32). It follows from Lemma 2.2 that the system (3.32) is consistent if and only if

$$r \begin{pmatrix} C_n & A_n \\ B_n & 0 \end{pmatrix} = r(A_n) + r(B_n), \quad (n = 3, 4), \quad (3.41)$$

$$r \begin{pmatrix} A_4 C_3 + C_4 B_3 & A_4 A_3 \\ B_4 B_3 & 0 \end{pmatrix} = r(A_4 A_3) + r(B_4 B_3), \quad (3.42)$$

or

$$R_{A_3} C_3 L_{B_3} = 0, \quad R_{A_{22}} C_{22} = 0, \quad C_{22} L_{B_{22}} = 0. \quad (3.43)$$

In this case, the general solution to the system (3.32) can be expressed as

$$X_3 = A_3^\dagger C_3 + U_2 B_3 + L_{A_3} T_1, \quad (3.44)$$

$$X_4 = -R_{A_3} C_3 B_3^\dagger + A_3 U_2 + V_2 R_{B_3}, \quad (3.45)$$

$$X_5 = -R_{(A_4 A_3)} (C_4 + A_4 R_{A_3} C_3 B_3^\dagger - A_4 V_2 R_{B_3}) B_4^\dagger + A_4 A_3 T_4 + T_5 R_{B_4}, \quad (3.46)$$

where

$$V_2 = A_{22}^\dagger C_{22} B_{22}^\dagger + L_{A_{22}} T_2 + T_3 R_{B_{22}}, \quad (3.47)$$

$$U_2 = (A_4 A_3)^\dagger (C_4 + A_4 R_{A_3} C_3 B_3^\dagger - A_4 V_2 R_{B_3}) + T_4 B_4 + L_{(A_4 A_3)} T_6, \quad (3.48)$$

and T_1, \dots, T_6 are arbitrary matrices over \mathbb{H} with appropriate sizes.

In the third step, equating X_3 in (3.36) and X_3 in (3.44) gives

$$\begin{aligned} & A_2^\dagger C_2 + (B_{11}^\dagger D_{11} C_{11}^\dagger + L_{B_{11}} W_2 + W_3 R_{C_{11}}) B_2 + L_{A_2} W_1 \\ &= A_3^\dagger C_3 + (A_4 A_3)^\dagger (C_4 + A_4 R_{A_3} C_3 B_3^\dagger) B_3 + T_4 B_4 B_3 + L_{(A_4 A_3)} T_6 B_3 + L_{A_3} T_1, \end{aligned}$$

i.e.,

$$A_{33} \begin{pmatrix} W_1 \\ T_1 \end{pmatrix} + (W_3, T_4) B_{33} + L_{B_{11}} W_2 B_2 + A_{44} T_6 B_3 = E_1. \quad (3.49)$$

Now we want to solve the matrix equation (3.49). It follows from Lemma 2.4 that the matrix equation (3.49) is consistent if and only if

$$R_M R_A E = 0, EL_B L_N = 0, R_A E L_D = 0, R_C E L_B = 0. \quad (3.50)$$

Hence, the general solution to the matrix equation (3.49) can be expressed as

$$\begin{pmatrix} W_1 \\ T_1 \end{pmatrix} = A_{33}^\dagger (E_1 - L_{B_{11}} W_2 B_2 - A_{44} T_6 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33}} Z_6, \quad (3.51)$$

$$(W_3, T_4) = R_{A_{33}} (E_1 - L_{B_{11}} W_2 B_2 - A_{44} T_6 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}, \quad (3.52)$$

$$W_2 = A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S Z_1 R_N D B^\dagger + L_A Z_2 + Z_3 R_B, \quad (3.53)$$

$$T_6 = M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D, \quad (3.54)$$

where Z_1, \dots, Z_8 are arbitrary matrices over \mathbb{H} with appropriate sizes.

Now we want to prove that (3.50) \iff (3.10)-(3.13). At first, we prove that $R_M R_A E = 0$ is equivalent to (3.10). Applying Lemma 2.5 to $R_M R_A E = 0$ gives

$$\begin{aligned} R_M R_A E = 0 &\Leftrightarrow r(R_A E, M) = r(M) \\ &\Leftrightarrow r(R_A E, R_A C) = r(R_A C) \\ &\Leftrightarrow r(A, C, E) = r(A, C) \\ &\Leftrightarrow r \begin{pmatrix} E_1 & L_{B_{11}} & A_{44} & A_{33} \\ B_{33} & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}}, A_{44}, A_{33}) + r(B_{33}) \\ &\Leftrightarrow r \begin{pmatrix} E_1 & L_{B_{11}} & L_{(A_4 A_3)} & L_{A_2} & L_{A_3} \\ R_{C_{11}} B_2 & 0 & 0 & 0 & 0 \\ B_4 B_3 & 0 & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}}, L_{(A_4 A_3)}, L_{A_2}, L_{A_3}) + r \begin{pmatrix} R_{C_{11}} B_2 \\ B_4 B_3 \end{pmatrix} \\ &\Leftrightarrow r \begin{pmatrix} E_1 & I & L_{(A_4 A_3)} \\ R_{C_{11}} B_2 & 0 & 0 \\ B_4 B_3 & 0 & 0 \\ 0 & B_{11} & 0 \end{pmatrix} = r \begin{pmatrix} I & L_{(A_4 A_3)} \\ B_{11} & 0 \end{pmatrix} + r \begin{pmatrix} R_{C_{11}} B_2 \\ B_4 B_3 \end{pmatrix} \\ &\Leftrightarrow r \begin{pmatrix} E_1 & I & I & 0 & 0 \\ B_2 & 0 & 0 & 0 & B_1 \\ B_4 B_3 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & A_1 & 0 \\ 0 & 0 & A_4 A_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} I & I & 0 \\ A_2 & 0 & A_1 \\ 0 & A_4 A_3 & 0 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 B_3 \end{pmatrix} \\ &\Leftrightarrow r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & A_4 C_3 + C_4 B_3 & 0 & A_4 A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_4 B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 B_3 \end{pmatrix} \\ &\Leftrightarrow (3.10). \end{aligned}$$

Similarly, we can show that $EL_B L_N = 0$, $R_A EL_D = 0$ and $R_C EL_B = 0$ are equivalent to (3.11), (3.12) and (3.13), respectively. \square

Next we give an example to illustrate Theorem 3.1.

Example 1. *Given the quaternion matrices:*

$$A_1 = \begin{pmatrix} 1+\mathbf{k} & 1+\mathbf{i}-\mathbf{k} & \mathbf{i}+\mathbf{j} \\ -1 & 2\mathbf{k} & 2+\mathbf{j}+\mathbf{k} \\ \mathbf{k} & 1+\mathbf{i}+\mathbf{k} & 2+\mathbf{i}+2\mathbf{j}+\mathbf{k} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2\mathbf{i}+\mathbf{k} & 0 & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ 2\mathbf{i}-\mathbf{j}+\mathbf{k} & -1+\mathbf{j} & 1+\mathbf{j} \\ \mathbf{j} & 1-\mathbf{j} & \mathbf{i}+\mathbf{k} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1+\mathbf{k} & 2 & \mathbf{i} \\ 2\mathbf{j} & 1-\mathbf{j} & -\mathbf{i}+\mathbf{k} \\ \mathbf{i}+\mathbf{j}+\mathbf{k} & 1 & \mathbf{k} \end{pmatrix}, \quad B_2 = \begin{pmatrix} -2+\mathbf{k} & \mathbf{i} & 1-\mathbf{j} \\ -\mathbf{j} & 1 & \mathbf{i}-\mathbf{k} \\ -2 & 1+\mathbf{i}+\mathbf{j} & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1+\mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{j} \\ \mathbf{i}-\mathbf{j} & -1-\mathbf{j} & \mathbf{k} \\ 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & -1+\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k} \end{pmatrix}, \quad B_3 = \begin{pmatrix} -1+\mathbf{j}+\mathbf{k} & 1+\mathbf{k} & \mathbf{i}+\mathbf{j} \\ 1-\mathbf{j}+\mathbf{k} & -1-\mathbf{j} & -\mathbf{i}+\mathbf{k} \\ 2\mathbf{k} & -\mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} \mathbf{j} & 1-\mathbf{j} & \mathbf{i}+\mathbf{k} \\ \mathbf{i} & -1+\mathbf{k} & \mathbf{j} \\ \mathbf{i}+\mathbf{j} & -\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1+\mathbf{k} & -1-\mathbf{k} & \mathbf{i}+\mathbf{j} \\ \mathbf{j} & \mathbf{i}+\mathbf{k} & 1 \\ 1+\mathbf{j}+\mathbf{k} & -1+\mathbf{i} & 1+\mathbf{i}+\mathbf{j} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -1-\mathbf{i}+5\mathbf{k} & -2-3\mathbf{j}+5\mathbf{k} & 3\mathbf{i}+5\mathbf{j} \\ -1-7\mathbf{i}+\mathbf{j}+2\mathbf{k} & 3\mathbf{j}+5\mathbf{k} & 4-8\mathbf{i}+7\mathbf{j} \\ -5-3\mathbf{i}+\mathbf{k} & -4-3\mathbf{i}+2\mathbf{j}+5\mathbf{k} & 1-5\mathbf{i}+9\mathbf{j}-4\mathbf{k} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 4\mathbf{i}+6\mathbf{j}+3\mathbf{k} & -2+5\mathbf{i}+4\mathbf{j} & -3-\mathbf{i}-\mathbf{j} \\ 2-\mathbf{i}+9\mathbf{j}-6\mathbf{k} & 6-5\mathbf{i}+5\mathbf{j}-4\mathbf{k} & -5+2\mathbf{i}-3\mathbf{j}+2\mathbf{k} \\ 4+\mathbf{i}+6\mathbf{j}+3\mathbf{k} & -2-\mathbf{i}+3\mathbf{j}-2\mathbf{k} & -2-\mathbf{i}+3\mathbf{j} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 7\mathbf{j}+6\mathbf{k} & 4+\mathbf{i}+\mathbf{j} & -1+2\mathbf{i}+6\mathbf{j} \\ -3-5\mathbf{i}+2\mathbf{j}+\mathbf{k} & -1+6\mathbf{j}+5\mathbf{k} & 2-\mathbf{i}+4\mathbf{j} \\ -8+2\mathbf{i}+3\mathbf{j}+6\mathbf{k} & 3+4\mathbf{i}+9\mathbf{j}-2\mathbf{k} & -2+\mathbf{i}+3\mathbf{j}-2\mathbf{k} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -1-2\mathbf{i}-\mathbf{j}-3\mathbf{k} & 3-3\mathbf{i}-2\mathbf{j}+\mathbf{k} & 1+2\mathbf{i} \\ 2-3\mathbf{i}-3\mathbf{j}-\mathbf{k} & -5-2\mathbf{i}+\mathbf{j}+\mathbf{k} & 1-2\mathbf{i}-2\mathbf{j} \\ 1-5\mathbf{i}-4\mathbf{j}-4\mathbf{k} & -2-5\mathbf{i}-\mathbf{j}+2\mathbf{k} & 2-2\mathbf{j} \end{pmatrix}.$$

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6). Check that

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 4, & \text{if } i = 1 \\ 6, & \text{if } i = 2 \\ 3, & \text{if } i = 3 \\ 4, & \text{if } i = 4 \end{cases},$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2) = 6,$$

$$r \begin{pmatrix} A_4C_3 + C_4B_3 & A_4A_3 \\ B_4B_3 & 0 \end{pmatrix} = r(A_4A_3) + r(B_4B_3) = 3,$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & A_4C_3 + C_4B_3 & 0 & A_4A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_4B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_4B_3 \end{pmatrix} = 9,$$

$$r \begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} = 6,$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ C_3 & 0 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} = 9.$$

$$r \begin{pmatrix} C_2 & A_2 \\ A_4C_3 + C_4B_3 & A_4A_3 \\ B_2 & 0 \\ B_4B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_4B_3 \end{pmatrix} = 6.$$

All the rank equalities in (3.7)-(3.13) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) is consistent. Note that

$$X_1 = \begin{pmatrix} 2 + \mathbf{j} + \mathbf{k} & 1 - 2\mathbf{i} + \mathbf{k} & \mathbf{i} - 2\mathbf{k} \\ -2 & \mathbf{i} + \mathbf{j} & 1 + \mathbf{i} + 2\mathbf{j} \\ \mathbf{j} + \mathbf{k} & 1 - \mathbf{i} + \mathbf{j} + \mathbf{k} & 1 + 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \end{pmatrix} \quad X_2 = \begin{pmatrix} \mathbf{i} + \mathbf{j} & -\mathbf{i} - \mathbf{j} & \mathbf{k} \\ 1 + 2\mathbf{j} & \mathbf{i} + \mathbf{j} + \mathbf{k} & -1 + \mathbf{i} + \mathbf{j} - \mathbf{k} \\ 2 + \mathbf{k} & -\mathbf{k} & 1 + \mathbf{j} + 2\mathbf{k} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 2\mathbf{i} + \mathbf{k} & 1 + 3\mathbf{i} - \mathbf{k} & \mathbf{j} \\ 1 + \mathbf{j} & -1 & \mathbf{i} - \mathbf{j} \\ 1 + 2\mathbf{i} + \mathbf{j} + \mathbf{k} & 3\mathbf{i} - \mathbf{k} & \mathbf{i} \end{pmatrix} \quad X_4 = \begin{pmatrix} -2 + \mathbf{j} & -1 + 2\mathbf{i} & \mathbf{k} \\ 0 & 1 + \mathbf{k} & \mathbf{i} + \mathbf{j} \\ -2 + \mathbf{k} & \mathbf{j} - \mathbf{k} & 1 \end{pmatrix},$$

and

$$X_5 = \begin{pmatrix} -2 + \mathbf{j} & -1 + 2\mathbf{i} & \mathbf{k} \\ 0 & 1 + \mathbf{k} & \mathbf{i} + \mathbf{j} \\ -2 + \mathbf{k} & \mathbf{j} - \mathbf{k} & 1 \end{pmatrix}$$

satisfy the system (1.6).

Let A_4, B_4 , and C_4 vanish in Theorem 3.1. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations (1.3).

Corollary 3.2. [35] *Let A_i, B_i , and $C_i (i = 1, 2, 3)$ be given. Set*

$$\begin{aligned} A_4 &= A_2 L_{A_3}, B_4 = R_{B_1} B_2, A = R_{A_4} A_2, B = B_3 L_{B_4}, \\ C &= R_{A_4} A_1, D = B_2 L_{B_4}, M = R_A C, N = D L_B, S = C L_M, \\ C_4 &= C_2 - A_2 A_3^\dagger C_3 - R_{A_1} C_1 B_1^\dagger B_2, E = R_{A_4} C_4 L_{B_4}. \end{aligned}$$

Then the following statements are equivalent:

(1) *The system of coupled generalized Sylvester real quaternion matrix equations (1.3) is consistent.*

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), (i = 1, 2, 3),$$

$$r \begin{pmatrix} A_1 & A_2 & C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{pmatrix} = r(A_1 \ A_2) + r(B_1 \ B_2),$$

$$r \begin{pmatrix} B_2 & 0 \\ B_3 & 0 \\ C_2 & A_2 \\ C_3 & A_3 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_3 \end{pmatrix},$$

$$r \begin{pmatrix} C_2 & C_1 & A_1 & A_2 \\ C_3 & 0 & 0 & A_3 \\ B_2 & B_1 & 0 & 0 \\ B_3 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r \begin{pmatrix} B_2 & B_1 \\ B_3 & 0 \end{pmatrix}.$$

(3)

$$R_{A_i} C_i L_{B_i} = 0, (i = 1, 2), R_M R_A E = 0,$$

$$E L_B L_N = 0, R_A E L_D = 0, R_C E L_B = 0.$$

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations (1.3) can be expressed as

$$X = A_1^\dagger C_1 - U_1 B_1 - L_{A_1} U_2, \ Y = -R_{A_1} C_1 B_1^\dagger - A_1 U_1 - U_3 R_{B_1},$$

$$Z = A_3^\dagger C_3 + V_1 B_3 + L_{A_3} V_2, \ W = -R_{A_3} C_3 B_3^\dagger + A_3 V_1 + V_3 R_{B_3},$$

where

$$V_2 = A_4^\dagger (C_4 - A_2 V_1 B_3 - A_1 U_1 B_2) - A_4^\dagger T_7 B_4 + L_{A_4} T_6,$$

$$U_3 = R_{A_4} (C_4 - A_2 V_1 B_3 - A_1 U_1 B_2) B_4^\dagger + A_4 A_4^\dagger T_7 + T_8 R_{B_4},$$

$$V_1 = A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S T_2 R_N D B^\dagger + L_A T_4 + T_5 R_B,$$

$$U_1 = M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D,$$

and $U_2, V_3, T_1, \dots, T_8$ are arbitrary matrices over \mathbb{H} with appropriate sizes.

4. Some solvability conditions and the general solution to system (1.7)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7). For simplicity, put

$$A_{11} = R_{(A_2 A_1)} A_2, \quad B_{11} = R_{B_1} L_{B_2}, \quad C_{11} = R_{(A_2 A_1)} (A_2 R_{A_1} C_1 B_1^\dagger + C_2) L_{B_2},$$

$$A_{22} = R_{B_3} B_4, \quad B_{22} = R_{A_4} A_3, \quad C_{22} = B_4 L_{A_{22}}, \quad D_{22} = R_{A_4} (R_{A_3} C_3 B_3^\dagger B_4 - C_4) L_{A_{22}},$$

$$A_{33} = (A_2 A_1, -L_{A_3}), \quad B_{33} = \begin{pmatrix} R_{B_2} \\ -R_{C_{22}} B_3 \end{pmatrix}, \quad A_{44} = R_{B_{11}} R_{B_1} B_2^\dagger, \quad B_{44} = -L_{B_{22}},$$

$$E_1 = A_3^\dagger C_3 + B_{22}^\dagger D_{22} C_{22}^\dagger B_3 + R_{(A_2 A_1)} C_2 B_2^\dagger + A_{11} R_{A_1} C_1 B_1^\dagger B_2^\dagger - C_{11} B_{11}^\dagger R_{B_1} B_2^\dagger,$$

$$A = R_{A_{33}} A_{11}, \quad B = A_{44} L_{B_{33}}, \quad C = R_{A_{33}} B_{44}, \quad D = B_3 L_{B_{33}},$$

$$E = R_{A_{33}} E_1 L_{B_{33}}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M.$$

Now we give the fundamental theorem of this section.

Theorem 4.1. *Let A_i, B_i , and $C_i (i = 1, 2, 3, 4)$ be given. Then the following statements are equivalent:*

(1) *The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) is consistent.*

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \quad (4.1)$$

$$r \begin{pmatrix} A_2 C_1 + C_2 B_1 & A_2 A_1 \\ B_2 B_1 & 0 \end{pmatrix} = r(A_2 A_1) + r(B_2 B_1), \quad (4.2)$$

$$r \begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r(A_3, A_4) + r(B_3, B_4), \quad (4.3)$$

$$r \begin{pmatrix} A_3 C_2 + C_3 B_2 & C_4 & A_3 A_2 & A_4 \\ B_3 B_2 & B_4 & 0 & 0 \end{pmatrix} = r(A_3 A_2, A_4) + r(B_3 B_2, B_4), \quad (4.4)$$

$$r \begin{pmatrix} A_3 A_2 C_1 + A_3 C_2 B_1 + C_3 B_2 B_1 & A_3 A_2 A_1 \\ B_3 B_2 B_1 & 0 \end{pmatrix} = r(A_3 A_2 A_1) + r(B_3 B_2 B_1), \quad (4.5)$$

$$r \begin{pmatrix} A_3C_2 + C_3B_2 & A_3A_2 \\ B_3B_2 & 0 \end{pmatrix} = r(A_3A_2) + r(B_3B_2), \quad (4.6)$$

$$r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & C_4 & A_4 & A_3A_2A_1 \\ B_3B_2B_1 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2A_1, A_4) + r(B_3B_2B_1, B_4). \quad (4.7)$$

(3)

$$R_{A_1}C_1L_{B_1} = 0, \quad R_{A_{11}}C_{11} = 0, \quad C_{11}L_{B_{11}} = 0,$$

$$R_{A_3}C_3L_{B_3} = 0, \quad R_{B_{22}}D_{22} = 0, \quad D_{22}L_{C_{22}} = 0,$$

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1B_1 + L_{A_1}W_1, \quad X_2 = -R_{A_1}C_1B_1^\dagger + A_1U_1 + V_1R_{B_1},$$

$$X_4 = -R_{A_3}C_3B_3^\dagger + A_3U_2 + V_2R_{B_3},$$

$$X_5 = A_4^\dagger(C_4 - R_{A_3}C_3B_3^\dagger B_4 + A_3U_1B_4) + T_4A_{22} + L_{A_4}T_6,$$

$$X_3 = -R_{(A_2A_1)}(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1})B_2^\dagger + A_2A_1W_4 + W_5R_{B_2},$$

or

$$X_3 = A_3^\dagger C_3 + U_2B_3 + L_{A_3}T_1,$$

where

$$V_1 = A_{11}^\dagger C_{11}B_{11}^\dagger + L_{A_{11}}W_2 + W_3R_{B_{11}},$$

$$U_1 = (A_2A_1)^\dagger(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1}) + W_4B_2 + L_{(A_2A_1)}W_6,$$

$$U_2 = B_{22}^\dagger D_{22}C_{22}^\dagger + L_{B_{22}}T_2 + T_3R_{C_{22}},$$

$$V_2 = -R_{A_4}(C_4 - R_{A_3}C_3B_3^\dagger B_4 + A_3U_2B_4)A_{22}^\dagger + A_4T_4 + T_5R_{A_{22}},$$

$$W_4 = (I_{p_1}, 0)[A_{33}^\dagger(E_1 - A_{11}W_3A_{44} - B_{44}T_2B_3) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$T_1 = (0, I_{p_2})[A_{33}^\dagger(E_1 - A_{11}W_3A_{44} - B_{44}T_2B_3) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$W_5 = [R_{A_{33}}(E_1 - A_{11}W_3A_{44} - B_{44}T_2B_3)B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix},$$

$$T_3 = [R_{A_{33}}(E_1 - A_{11}W_3A_{44} - B_{44}T_2B_3)B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix},$$

$$W_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1 R_N DB^\dagger + L_A Z_2 + Z_3 R_B,$$

$$T_2 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D,$$

the remaining W_j, T_j, Z_j are arbitrary matrices over \mathbb{H} , p_1 and p_2 are the column numbers of A_1 and A_3 , respectively, p_3 and p_4 are the row numbers of B_2 and B_4 , respectively.

Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) into two parts

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \end{cases} \quad (4.8)$$

and

$$\begin{cases} A_3 X_3 - X_4 B_3 = C_3, \\ A_4 X_5 - X_4 B_4 = C_4. \end{cases} \quad (4.9)$$

Applying the main idea of Theorem 3.1, Lemma 2.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove Theorem 4.1. \square

Now we give an example to illustrate Theorem 4.1.

Example 2. Given the quaternion matrices:

$$\begin{aligned} A_1 &= \begin{pmatrix} \mathbf{i} & \mathbf{j} & 1+\mathbf{k} \\ \mathbf{k} & \mathbf{i}+\mathbf{j}-2\mathbf{k} & -2+\mathbf{k} \\ 1+\mathbf{i}+\mathbf{j} & 2-\mathbf{i} & -\mathbf{j} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & \mathbf{k} & \mathbf{i}+\mathbf{k} \\ \mathbf{i} & \mathbf{k} & -1+\mathbf{k} \\ 1+\mathbf{i} & 2\mathbf{k} & -1+\mathbf{i}+2\mathbf{k} \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 1+\mathbf{i}+\mathbf{k} & 1-\mathbf{i} & 1+\mathbf{j}+\mathbf{k} \\ -1+\mathbf{i}-\mathbf{j} & 1+\mathbf{i} & \mathbf{i}-\mathbf{j}+\mathbf{k} \\ 2\mathbf{i}-\mathbf{j}+\mathbf{k} & 2 & 1+\mathbf{i}+2\mathbf{k} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1+\mathbf{j}-\mathbf{k} & -1+2\mathbf{j}-\mathbf{k} & 2 \\ \mathbf{i}+\mathbf{j} & 1+\mathbf{k} & 1+\mathbf{i}-\mathbf{j} \\ 1+\mathbf{i}+2\mathbf{j}-\mathbf{k} & 2\mathbf{j} & 3+\mathbf{i}-\mathbf{j} \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 2\mathbf{j} & 1-\mathbf{j} & \mathbf{i}+\mathbf{k} \\ \mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{j} & -1 \\ \mathbf{i}+3\mathbf{j}+\mathbf{k} & 2 & -1+\mathbf{i}+\mathbf{k} \end{pmatrix}, \quad B_3 = \begin{pmatrix} -1 & 1+\mathbf{j} & \mathbf{i}-\mathbf{j} \\ -\mathbf{i} & \mathbf{i}+2\mathbf{j} & -1-2\mathbf{j} \\ -1-\mathbf{i} & 1+\mathbf{i}+\mathbf{k} & -1+\mathbf{i}-\mathbf{k} \end{pmatrix}, \\ A_4 &= \begin{pmatrix} 2+\mathbf{i}+\mathbf{j}-2\mathbf{k} & 1-2\mathbf{i}-\mathbf{j} & \mathbf{i} \\ 1+\mathbf{j}-2\mathbf{k} & 0 & -\mathbf{i}+\mathbf{j} \\ 3+\mathbf{i} & 1-2\mathbf{i}+\mathbf{k} & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} \mathbf{j} & 1-\mathbf{j} & \mathbf{i}+\mathbf{k} \\ \mathbf{i} & 1+\mathbf{j} & -\mathbf{k} \\ \mathbf{i}+\mathbf{j} & 2 & \mathbf{i} \end{pmatrix}, \\ C_1 &= \begin{pmatrix} 1+\mathbf{i}-\mathbf{j}-2\mathbf{k} & \mathbf{i}-2\mathbf{j}+6\mathbf{k} & 1+\mathbf{i}-2\mathbf{j}+4\mathbf{k} \\ \mathbf{i}-4\mathbf{j}+3\mathbf{k} & 3-2\mathbf{i}+3\mathbf{j}-6\mathbf{k} & 1-\mathbf{i}-2\mathbf{j}+\mathbf{k} \\ 4\mathbf{i}+\mathbf{j}-6\mathbf{k} & 1-2\mathbf{i}-4\mathbf{j}+8\mathbf{k} & 2-7\mathbf{j}+7\mathbf{k} \end{pmatrix}, \\ C_2 &= \begin{pmatrix} 3-3\mathbf{i}+3\mathbf{j}+3\mathbf{k} & -2-\mathbf{i}+\mathbf{j}-9\mathbf{k} & -6-6\mathbf{i}-3\mathbf{j}+8\mathbf{k} \\ -2-8\mathbf{j}+8\mathbf{k} & \mathbf{i}+2\mathbf{j} & -4-6\mathbf{i}-5\mathbf{j}-6\mathbf{k} \\ 1-3\mathbf{i}-5\mathbf{j}+11\mathbf{k} & -2+3\mathbf{j}-9\mathbf{k} & -10-12\mathbf{i}-8\mathbf{j}+2\mathbf{k} \end{pmatrix}, \end{aligned}$$

$$C_3 = \begin{pmatrix} 3 - 4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} & -2 + 6\mathbf{i} + 2\mathbf{j} - \mathbf{k} & 3 - \mathbf{i} - \mathbf{j} \\ 4 - \mathbf{i} + 6\mathbf{j} - \mathbf{k} & -4 - \mathbf{i} - 4\mathbf{j} + 10\mathbf{k} & -2 - 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} \\ 1 - 4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} & -1 + 8\mathbf{i} + 3\mathbf{k} & 3 - \mathbf{i} + 3\mathbf{j} - \mathbf{k} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 2 + \mathbf{i} - 3\mathbf{k} & 2 + 2\mathbf{j} + 3\mathbf{k} & 8 - 2\mathbf{k} \\ -1 + 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} & -3 + 6\mathbf{i} - 4\mathbf{j} + \mathbf{k} & -1 - 3\mathbf{i} + 3\mathbf{j} \\ 1 - 3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k} & 1 - 3\mathbf{j} + 2\mathbf{k} & 4 - 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{pmatrix}.$$

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7). Check that

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 5, & \text{if } i = 1 \\ 3, & \text{if } i = 2 \\ 4, & \text{if } i = 3 \\ 5, & \text{if } i = 4 \end{cases},$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1) = 3,$$

$$r \begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r(A_3, A_4) + r(B_3, B_4) = 6,$$

$$r \begin{pmatrix} A_3C_2 + C_3B_2 & C_4 & A_3A_2 & A_4 \\ B_3B_2 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2, A_4) + r(B_3B_2, B_4) = 6,$$

$$r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & A_3A_2A_1 \\ B_3B_2B_1 & 0 \end{pmatrix} = r(A_3A_2A_1) + r(B_3B_2B_1) = 3,$$

$$r \begin{pmatrix} A_3C_2 + C_3B_2 & A_3A_2 \\ B_3B_2 & 0 \end{pmatrix} = r(A_3A_2) + r(B_3B_2) = 3,$$

$$r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & C_4 & A_4 & A_3A_2A_1 \\ B_3B_2B_1 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2A_1, A_4) + r(B_3B_2B_1, B_4) = 6.$$

All the rank equalities in (4.1)-(4.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) is consistent. Note that

$$X_1 = \begin{pmatrix} 1 - \mathbf{k} & \mathbf{i} + \mathbf{j} + 2\mathbf{k} & \mathbf{k} \\ -1 + \mathbf{j} & -\mathbf{i} - 2\mathbf{j} + \mathbf{k} & -\mathbf{j} \\ \mathbf{j} - \mathbf{k} & -\mathbf{j} + 3\mathbf{k} & -\mathbf{j} + \mathbf{k} \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 + \mathbf{j} & \mathbf{i} & -1 + \mathbf{k} \\ \mathbf{i} + \mathbf{j} + \mathbf{k} & -1 & -\mathbf{i} - \mathbf{j} + \mathbf{k} \\ 1 + \mathbf{i} + 2\mathbf{j} + \mathbf{k} & -1 + \mathbf{i} & -1 - \mathbf{i} - \mathbf{j} + 2\mathbf{k} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -2\mathbf{k} & 2 + \mathbf{k} & \mathbf{i} \\ \mathbf{i} + 2\mathbf{j} & 1 - \mathbf{j} & 1 - \mathbf{i} \\ \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} & 3 - \mathbf{j} + \mathbf{k} & 1 \end{pmatrix} \quad X_4 = \begin{pmatrix} \mathbf{k} & \mathbf{i} & 1 - \mathbf{k} \\ 1 - 2\mathbf{i} + \mathbf{j} - \mathbf{k} & 1 - 3\mathbf{i} & 1 + \mathbf{i} + 2\mathbf{j} + \mathbf{k} \\ -1 & 2 + \mathbf{k} & \mathbf{i} \end{pmatrix},$$

and

$$X_5 = \begin{pmatrix} \mathbf{i} + \mathbf{j} & \mathbf{k} & 1 + \mathbf{k} \\ 1 + 2\mathbf{j} & \mathbf{i} & 1 + \mathbf{j} \\ \mathbf{j} + \mathbf{k} & \mathbf{k} & 1 + 2\mathbf{j} \end{pmatrix}$$

satisfy the system (1.7).

Let A_4, B_4 , and C_4 vanish in Theorem 4.1. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations (1.4).

Corollary 4.2. [35] *Let A_i, B_i , and $C_i (i = 1, 2, 3)$ be given. Set*

$$\begin{aligned} A &= R_{(A_2 A_1)} A_2, B = R_{B_1} L_{(B_3 B_2)}, C = R_{(A_2 A_1)} L_{A_3}, \\ D &= B_2 L_{(B_3 B_2)}, M = R_A C, N = D L_B, S = C L_M, \\ C_4 &= C_2 + A_3^\dagger C_3 B_2 + A_2 R_{A_1} C_1 B_1^\dagger, E = R_{(A_2 A_1)} C_4 L_{(B_3 B_2)}. \end{aligned}$$

Then the following statements are equivalent:

(1) *The system of coupled generalized Sylvester real quaternion matrix equations (1.4) is consistent.*

(2)

$$\begin{aligned} r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} &= r(A_i) + r(B_i), (i = 1, 2, 3), \\ r \begin{pmatrix} A_3 C_2 + C_3 B_2 & A_3 A_2 \\ B_3 B_2 & 0 \end{pmatrix} &= r(A_3 A_2) + r(B_3 B_2), \\ r \begin{pmatrix} A_2 C_1 + C_2 B_1 & A_2 A_1 \\ B_2 B_1 & 0 \end{pmatrix} &= r(A_2 A_1) + r(B_2 B_1), \\ r \begin{pmatrix} A_3 A_2 C_1 + A_3 C_2 B_1 + C_3 B_2 B_1 & A_3 A_2 A_1 \\ B_3 B_2 B_1 & 0 \end{pmatrix} &= r(A_3 A_2 A_1) + r(B_3 B_2 B_1). \end{aligned}$$

(3)

$$\begin{aligned} R_{A_i} C_i L_{B_i} &= 0, (i = 1, 2), R_M R_A E = 0, \\ E L_B L_N &= 0, R_A E L_D = 0, R_C E L_B = 0. \end{aligned}$$

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations (1.4) can be expressed as

$$\begin{aligned} X &= A_1^\dagger C_1 + U_1 B_1 + L_{A_1} U_2, \quad Y = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + U_3 R_{B_1}, \\ Z &= A_3^\dagger C_3 - V_1 B_3 - L_{A_3} V_2, \quad W = -R_{A_3} C_3 B_3^\dagger - A_3 V_1 - V_3 R_{B_3}, \end{aligned}$$

where

$$U_1 = (A_2 A_1)^\dagger (C_4 - A_2 U_3 R_{B_1} - L_{A_3} V_2 B_2) - (A_2 A_1)^\dagger T_7 (B_3 B_2) + L_{(A_2 A_1)} T_6,$$

$$V_1 = R_{(A_2A_1)}(C_4 - A_2U_3R_{B_1} - L_{A_3}V_2B_2)(B_3B_2)^\dagger + (A_2A_1)(A_2A_1)^\dagger T_7 + T_8R_{(B_3B_2)},$$

$$U_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger ST_2R_NDB^\dagger + L_AT_4 + T_5R_B,$$

$$V_2 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

and $U_2, V_3, T_1, \dots, T_8$ are arbitrary matrices over \mathbb{H} with appropriate sizes.

5. Some solvability conditions and the general solution to system (1.8)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8). For simplicity, put

$$A_{11} = R_{(A_2A_1)}A_2, \quad B_{11} = R_{B_1}L_{B_2}, \quad C_{11} = R_{(A_2A_1)}(A_2R_{A_1}C_1B_1^\dagger + C_2)L_{B_2},$$

$$A_{22} = R_{(A_3A_4)}A_3, \quad B_{22} = R_{B_4}L_{B_3}, \quad C_{22} = R_{(A_3A_4)}(A_3R_{A_4}C_4B_4^\dagger + C_3)L_{B_3},$$

$$A_{33} = (A_2A_1, -A_3A_4), \quad B_{33} = \begin{pmatrix} R_{B_2} \\ -R_{B_3} \end{pmatrix}, \quad A_{44} = R_{B_{11}}R_{B_1}B_2^\dagger, \quad B_{44} = R_{B_{22}}R_{B_4}B_3^\dagger,$$

$$E_1 = R_{(A_2A_1)}C_2B_2^\dagger + A_{11}R_{A_1}C_1B_1^\dagger B_2^\dagger - C_{11}B_{11}^\dagger R_{B_1}B_2^\dagger - \\ R_{(A_3A_4)}C_3B_3^\dagger - A_{22}R_{A_4}C_4B_4^\dagger B_2^\dagger + C_{22}B_{22}^\dagger R_{B_4}B_3^\dagger,$$

$$A = R_{A_{33}}A_{11}, \quad B = A_{44}L_{B_{33}}, \quad C = -R_{A_{33}}A_{22}, \quad D = B_{44}L_{B_{33}},$$

$$E = R_{A_{33}}E_1L_{B_{33}}, \quad M = R_AC, \quad N = DL_B, \quad S = CL_M.$$

Now we give the fundamental theorem of this section.

Theorem 5.1. *Let A_i, B_i , and $C_i (i = 1, 2, 3, 4)$ be given. Then the following statements are equivalent:*

(1) *The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) is consistent.*

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \quad (5.1)$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1), \quad (5.2)$$

$$r \begin{pmatrix} A_3C_4 + C_3B_4 & A_3A_4 \\ B_3B_4 & 0 \end{pmatrix} = r(A_3A_4) + r(B_3B_4), \quad (5.3)$$

$$r \begin{pmatrix} C_2 & C_3 & A_2 & A_3 \\ B_2 & B_3 & 0 & 0 \end{pmatrix} = r(A_2, A_3) + r(B_2, B_3), \quad (5.4)$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_3C_4 + C_3B_4 & A_2A_1 & A_3A_4 \\ B_2B_1 & B_3B_4 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3A_4) + r(B_2B_1, B_3B_4), \quad (5.5)$$

$$r \begin{pmatrix} C_2 & A_3C_4 + C_3B_4 & A_2 & A_3A_4 \\ B_2 & B_3B_4 & 0 & 0 \end{pmatrix} = r(A_2, A_3A_4) + r(B_2, B_3B_4), \quad (5.6)$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & C_3 & A_2A_1 & A_3 \\ B_2B_1 & B_3 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3) + r(B_2B_1, B_3). \quad (5.7)$$

(3)

$$R_{A_1}C_1L_{B_1} = 0, \quad R_{A_{11}}C_{11} = 0, \quad C_{11}L_{B_{11}} = 0,$$

$$R_{A_4}C_4L_{B_4} = 0, \quad R_{A_{22}}C_{22} = 0, \quad C_{22}L_{B_{22}} = 0,$$

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1B_1 + L_{A_1}W_1, \quad X_2 = -R_{A_1}C_1B_1^\dagger + A_1U_1 + V_1R_{B_1},$$

$$X_4 = -R_{A_4}C_4B_4^\dagger + A_4U_2 + V_2R_{B_4}, \quad X_5 = A_4^\dagger C_4 + U_2B_4 + L_{A_4}T_1,$$

$$X_3 = -R_{(A_2A_1)}(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1})B_2^\dagger + A_2A_1W_4 + W_5R_{B_2},$$

or

$$X_3 = -R_{(A_3A_4)}(C_3 + A_3R_{A_4}C_4B_4^\dagger - A_3V_2R_{B_4})B_3^\dagger + A_3A_4T_4 + T_5R_{B_3},$$

where

$$V_1 = A_{11}^\dagger C_{11}B_{11}^\dagger + L_{A_{11}}W_2 + W_3R_{B_{11}},$$

$$U_1 = (A_2A_1)^\dagger(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1}) + W_4B_2 + L_{(A_2A_1)}W_6,$$

$$V_2 = A_{22}^\dagger C_{22}B_{22}^\dagger + L_{A_{22}}T_2 + T_3R_{B_{22}},$$

$$U_2 = (A_3A_4)^\dagger(C_3 + A_3R_{A_4}C_4B_4^\dagger - A_3V_2R_{B_4}) + T_4B_3 + L_{(A_3A_4)}T_6,$$

$$W_4 = (I_{p_1}, 0)[A_{33}^\dagger(E_1 - A_{11}W_3A_{44} + A_{22}T_3B_{44}) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$T_4 = (0, I_{p_2})[A_{33}^\dagger(E_1 - A_{11}W_3A_{44} + A_{22}T_3B_{44}) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$W_5 = [R_{A_{33}}(E_1 - A_{11}W_3A_{44} + A_{22}T_3B_{44})B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix},$$

$$T_5 = [R_{A_{33}}(E_1 - A_{11}W_3A_{44} + A_{22}T_3B_{44})B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix},$$

$$W_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1 R_N DB^\dagger + L_A Z_2 + Z_3 R_B,$$

$$T_3 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D,$$

the remaining W_j, T_j, Z_j are arbitrary matrices over \mathbb{H} , p_1 and p_2 are the column numbers of A_1 and A_4 , respectively, p_3 and p_4 are the row numbers of B_2 and B_3 , respectively.

Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) into two parts

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \end{cases} \quad (5.8)$$

and

$$\begin{cases} A_3 X_4 - X_3 B_3 = C_3, \\ A_4 X_5 - X_4 B_4 = C_4. \end{cases} \quad (5.9)$$

Applying the main idea of Theorem 3.1, Lemma 2.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove Theorem 5.1. \square

Now we give an example to illustrate Theorem 5.1.

Example 3. Given the quaternion matrices:

$$A_1 = \begin{pmatrix} \mathbf{i} + \mathbf{j} + \mathbf{k} & 2 + \mathbf{i} + \mathbf{j} - \mathbf{k} \\ -1 + \mathbf{j} + \mathbf{k} & -1 + 2\mathbf{i} + \mathbf{j} - \mathbf{k} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 + \mathbf{k} & \mathbf{j} - \mathbf{k} \\ \mathbf{i} + 2\mathbf{k} & 2\mathbf{j} - 2\mathbf{k} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \mathbf{i} & 2 + \mathbf{j} \\ 1 + \mathbf{i} + \mathbf{k} & -\mathbf{j} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 + \mathbf{j} + 2\mathbf{k} & \mathbf{i} + 3\mathbf{k} \\ \mathbf{j} & 1 + \mathbf{i} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 + \mathbf{k} & \mathbf{i} + \mathbf{k} \\ 1 + \mathbf{i} + \mathbf{j} + \mathbf{k} & -1 + \mathbf{i} + \mathbf{j} + \mathbf{k} \end{pmatrix}, \quad B_3 = \begin{pmatrix} 2 + \mathbf{k} & -\mathbf{i} - 2\mathbf{j} \\ 2 - 2\mathbf{i} - \mathbf{k} & -1 - \mathbf{i} + 2\mathbf{j} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 3\mathbf{i} + \mathbf{j} & 1 + 2\mathbf{j} \\ 2 + \mathbf{k} & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} -\mathbf{j} & 1 + 2\mathbf{j} \\ -\mathbf{k} & \mathbf{i} + 2\mathbf{k} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 - 5\mathbf{i} + \mathbf{j} + \mathbf{k} & 3 + 8\mathbf{i} - 7\mathbf{j} - 3\mathbf{k} \\ 3 + 2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k} & -8 + 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 + \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} & 5 + 3\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} \\ 1 + 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} & -3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} -7\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} & -3 - 7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \\ -3 + \mathbf{i} - 3\mathbf{j} - \mathbf{k} & -6 - 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} \end{pmatrix}, \quad C_4 = \begin{pmatrix} -2 + 7\mathbf{i} - 4\mathbf{j} & 2 - 7\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} \\ -6 + \mathbf{i} - \mathbf{j} + \mathbf{k} & 13 - 3\mathbf{i} + \mathbf{j} + 4\mathbf{k} \end{pmatrix}.$$

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8). Check that

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 4, & \text{if } i = 1, 2, 3 \\ 3, & \text{if } i = 4 \end{cases},$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1) = 4,$$

$$r \begin{pmatrix} A_3C_4 + C_3B_4 & A_3A_4 \\ B_3B_4 & 0 \end{pmatrix} = r(A_3A_4) + r(B_3B_4) = 3,$$

$$r \begin{pmatrix} C_2 & C_3 & A_2 & A_3 \\ B_2 & B_3 & 0 & 0 \end{pmatrix} = r(A_2, A_3) + r(B_2, B_3) = 4,$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_3C_4 + C_3B_4 & A_2A_1 & A_3A_4 \\ B_2B_1 & B_3B_4 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3A_4) + r(B_2B_1, B_3B_4) = 4,$$

$$r \begin{pmatrix} C_2 & A_3C_4 + C_3B_4 & A_2 & A_3A_4 \\ B_2 & B_3B_4 & 0 & 0 \end{pmatrix} = r(A_2, A_3A_4) + r(B_2, B_3B_4) = 4,$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & C_3 & A_2A_1 & A_3 \\ B_2B_1 & B_3 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3) + r(B_2B_1, B_3) = 4.$$

All the rank equalities in (5.1)-(5.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) is consistent. Note that

$$X_1 = \begin{pmatrix} \mathbf{i} + \mathbf{j} & -1 + \mathbf{k} \\ 2 + \mathbf{k} & 2\mathbf{i} - \mathbf{j} \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 + 2\mathbf{i} + \mathbf{j} & -\mathbf{i} + 2\mathbf{j} \\ \mathbf{k} & 1 + 2\mathbf{k} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} \mathbf{i} & -1 + \mathbf{j} \\ -1 & -\mathbf{i} + \mathbf{j} + \mathbf{k} \end{pmatrix} \quad X_4 = \begin{pmatrix} -1 + 2\mathbf{j} & 1 + 3\mathbf{j} \\ -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} & \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \end{pmatrix},$$

and

$$X_5 = \begin{pmatrix} \mathbf{k} & 1 + 2\mathbf{j} \\ \mathbf{i} + \mathbf{k} & 1 - \mathbf{i} + \mathbf{j} - \mathbf{k} \end{pmatrix}$$

satisfy the system (1.8).

Let A_4, B_4 , and C_4 vanish in Theorem 5.1. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations (1.5).

Corollary 5.2. [35] *Let A_i, B_i , and $C_i (i = 1, 2, 3)$ be given. Set*

$$\begin{aligned} A &= R_{(A_2A_1)}A_2, B = R_{B_1}L_{(R_{B_3}B_2)}, C = R_{(A_2A_1)}A_3, \\ D &= B_2L_{(R_{B_3}B_2)}, C_4 = C_2 + A_2^\dagger R_{A_1}C_1B_1^\dagger - R_{A_3}C_3B_3^\dagger B_2, \\ E &= R_{(A_2A_1)}C_4L_{(R_{B_3}B_2)}, M = R_A C, N = DL_B, S = CL_M. \end{aligned}$$

Then the following statements are equivalent:

(1) The system of coupled generalized Sylvester real quaternion matrix equations (1.5) is consistent.

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), (i = 1, 2, 3),$$

$$r \begin{pmatrix} A_2 & A_3 & C_2 & C_3 \\ 0 & 0 & B_2 & B_3 \end{pmatrix} = r(A_2, A_3) + r(B_2, B_3),$$

$$r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1),$$

$$r \begin{pmatrix} A_3 & A_2A_1 & C_3 & A_2C_1 + C_2B_1 \\ 0 & 0 & B_3 & B_2B_1 \end{pmatrix} = r(A_3, A_2A_1) + r(B_3, B_2B_1).$$

(3)

$$R_{A_i}C_iL_{B_i} = 0, (i = 1, 2), R_MR_AE = 0,$$

$$EL_B L_N = 0, R_AEL_D = 0, R_CEL_B = 0.$$

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations (1.5) can be expressed as

$$X = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} U_2, \quad Y = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + U_3 R_{B_1},$$

$$Z = -R_{A_3} C_3 B_3^\dagger - A_3 V_1 - V_3 R_{B_3}, \quad W = A_3^\dagger C_3 - V_1 B_3 - L_{A_3} V_2,$$

where

$$U_1 = (A_2 A_1)^\dagger (C_4 - A_2 U_3 R_{B_1} - A_3 V_1 B_2) - (A_2 A_1)^\dagger T_7 (R_{B_3} B_2) + L_{(A_2 A_1)} T_6,$$

$$V_3 = R_{(A_2 A_1)} (C_4 - A_2 U_3 R_{B_1} - A_3 V_1 B_2) (R_{B_3} B_2)^\dagger + (A_2 A_1) (A_2 A_1)^\dagger T_7 + T_8 R_{(R_{B_3} B_2)},$$

$$U_3 = A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S T_2 R_N D B^\dagger + L_A T_4 + T_5 R_B,$$

$$V_1 = M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D,$$

and $U_2, V_2, T_1, \dots, T_8$ are arbitrary matrices over \mathbb{H} with appropriate sizes.

6. Some solvability conditions and the general solution to system (1.9)

Our goal of this section is to give some necessary and sufficient conditions and the general solution to the system (1.9). Set

$$A_{jj} = R_{(A_{2j}A_{2j-1})}A_{2j}, B_{jj} = R_{B_{2j-1}}L_{B_{2j}}, C_{jj} = R_{(A_{2j}A_{2j-1})}(A_{2j}R_{A_{2j-1}}C_{2j-1}B_{2j-1}^\dagger + C_{2j})L_{B_{2j}},$$

$$(j = 1, 2), A_{33} = (A_2A_1, -L_{A_3}), B_{33} = \begin{pmatrix} R_{B_2} \\ -B_4B_3 \end{pmatrix}, A_{44} = R_{B_{11}}R_{B_1}B_2^\dagger, B_{44} = -L_{(A_4A_3)},$$

$$E_1 = A_3^\dagger C_3 + (A_4A_3)^\dagger C_4B_3 + (A_4A_3)^\dagger A_4R_{A_3}C_3 + R_{(A_2A_1)}C_2B_2^\dagger + A_{11}R_{A_1}C_1B_1^\dagger B_2^\dagger - C_{11}B_{11}^\dagger R_{B_1}B_2^\dagger,$$

$$A = R_{A_{33}}A_{11}, B = A_{44}L_{B_{33}}, C = R_{A_{33}}B_{44}, D = B_3L_{B_{33}},$$

$$M = R_AC, N = DL_B, S = CL_M, E = R_{A_{33}}E_1L_{B_{33}}.$$

Now we give the fundamental theorem of this section.

Theorem 6.1. *Let A_i, B_i , and $C_i (i = 1, 2, 3, 4)$ be given. Then the following statements are equivalent:*

(1) *The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) is consistent.*

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \quad (6.1)$$

$$r \begin{pmatrix} A_{k+1}C_k + C_{k+1}B_k & A_{k+1}A_k \\ B_{k+1}B_k & 0 \end{pmatrix} = r(A_{k+1}A_k) + r(B_{k+1}B_k), \quad (k = 1, 2, 3), \quad (6.2)$$

$$\begin{aligned} & r \begin{pmatrix} A_{j+2}A_{j+1}C_j + A_{j+2}C_{j+1}B_j + C_{j+2}B_{j+1}B_j & A_{j+2}A_{j+1}A_j \\ B_{j+2}B_{j+1}B_j & 0 \end{pmatrix} \\ & = r(A_{j+2}A_{j+1}A_j) + r(B_{j+2}B_{j+1}B_j), \quad (j = 1, 2), \end{aligned} \quad (6.3)$$

$$\begin{aligned} & r \begin{pmatrix} A_4A_3A_2C_1 + A_4A_3C_2A_1 + A_4C_3A_2A_1 + C_4A_3A_2A_1 & A_4A_3A_2A_1 \\ B_4B_3B_2B_1 & 0 \end{pmatrix} \\ & = r(A_4A_3A_2A_1) + r(B_4B_3B_2B_1). \end{aligned} \quad (6.4)$$

(3)

$$R_{A_{2j-1}}C_{2j-1}L_{B_{2j-1}} = 0, R_{A_{jj}}C_{jj} = 0, C_{jj}L_{B_{jj}} = 0, \quad (j = 1, 2),$$

$$R_MR_AE = 0, EL_BL_N = 0, R_AEL_D = 0, R_CEL_B = 0.$$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1B_1 + L_{A_1}W_1, X_2 = -R_{A_1}C_1B_1^\dagger + A_1U_1 + V_1R_{B_1},$$

$$X_3 = A_3^\dagger C_3 + U_2 B_3 + L_{A_3} T_1, \quad X_4 = -R_{A_3} C_3 B_3^\dagger + A_3 U_2 + V_2 R_{B_3},$$

$$X_3 = -R_{(A_2 A_1)} C_2 B_2^\dagger - A_{11} R_{A_1} C_1 B_1^\dagger B_2^\dagger + C_{11} B_{11}^\dagger R_{B_1} B_2^\dagger + A_{11} W_3 A_{44} + A_2 A_1 W_4 + W_5 R_{B_2},$$

or

$$X_3 = A_3^\dagger C_3 + (A_4 A_3)^\dagger C_4 B_3 + (A_4 A_3)^\dagger A_4 R_{A_3} C_3 + L_{A_3} T_1 + T_4 B_4 B_3 + L_{(A_4 A_3)} T_6 B_3,$$

where

$$V_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_2 + W_3 R_{B_{11}},$$

$$U_1 = (A_2 A_1)^\dagger (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_1)} W_6,$$

$$V_2 = A_{22}^\dagger C_{22} B_{22}^\dagger + L_{A_{22}} T_2 + T_3 R_{B_{22}},$$

$$U_2 = (A_4 A_3)^\dagger (C_4 + A_4 R_{A_3} C_3 B_3^\dagger - A_4 V_2 R_{B_3}) + T_4 B_4 + L_{(A_4 A_3)} T_6,$$

$$W_4 = (I_{p_1}, 0)[A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33}} Z_6],$$

$$T_1 = (0, I_{p_2})[A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33}} Z_6],$$

$$W_5 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix},$$

$$T_4 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix},$$

$$W_3 = A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S Z_1 R_N D B^\dagger + L_A Z_2 + Z_3 R_B,$$

$$T_6 = M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D,$$

the remaining W_j, T_j, Z_j are arbitrary matrices over \mathbb{H} , p_1 and p_2 are the column numbers of A_1 and A_3 , respectively, p_3 and p_4 are the row numbers of B_2 and B_4 , respectively.

Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) into two parts

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_2 - X_3 B_2 = C_2, \end{cases} \quad (6.5)$$

and

$$\begin{cases} A_3 X_3 - X_4 B_3 = C_3, \\ A_4 X_4 - X_5 B_4 = C_4. \end{cases} \quad (6.6)$$

Applying the main idea of Theorem 3.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove Theorem 6.1. \square

Now we give an example to illustrate Theorem 6.1.

Example 4. *Given the quaternion matrices:*

$$A_1 = \begin{pmatrix} 1+\mathbf{k} & -1 & 2\mathbf{i}+\mathbf{j} \\ 0 & \mathbf{i}+\mathbf{k} & \mathbf{i}-2\mathbf{j} \\ 1+\mathbf{i} & 2-\mathbf{i} & 1+\mathbf{k} \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1+\mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{j}+\mathbf{k} \\ -2-\mathbf{j} & 2\mathbf{i}-\mathbf{j} & -\mathbf{j}+\mathbf{k} \\ 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & -1+\mathbf{i}-\mathbf{j}+\mathbf{k} & 2\mathbf{k} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \mathbf{i} & \mathbf{j} & 1+2\mathbf{i}+\mathbf{k} \\ \mathbf{k} & \mathbf{i}-\mathbf{j} & -1-2\mathbf{j}+\mathbf{k} \\ \mathbf{i}+\mathbf{k} & \mathbf{i} & 2\mathbf{i}-2\mathbf{j}+2\mathbf{k} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \mathbf{j} & 1+2\mathbf{i}+\mathbf{j} & -\mathbf{i}+\mathbf{k} \\ \mathbf{i}-\mathbf{j} & \mathbf{k} & 1+2\mathbf{j} \\ \mathbf{i} & 1+2\mathbf{i}+\mathbf{j}+\mathbf{k} & 1-\mathbf{i}+2\mathbf{j}+\mathbf{k} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1+2\mathbf{i}+\mathbf{k} & 2-\mathbf{i}-\mathbf{k} & 1+\mathbf{j} \\ -1-2\mathbf{i}-\mathbf{j}+\mathbf{k} & -2+\mathbf{i}+\mathbf{j}-\mathbf{k} & -1+\mathbf{j}+\mathbf{k} \\ -\mathbf{k} & \mathbf{k} & -\mathbf{j} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \mathbf{i}+2\mathbf{j} & 1+3\mathbf{j} & \mathbf{j}-3\mathbf{k} \\ -1+\mathbf{i}-2\mathbf{j} & 1+\mathbf{i}-3\mathbf{j} & -\mathbf{j}+3\mathbf{k} \\ \mathbf{i} & 1 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2+3\mathbf{i}+\mathbf{k} & 3-\mathbf{j} & \mathbf{i}+\mathbf{j}+\mathbf{k} \\ -3+2\mathbf{i}-\mathbf{j} & 3\mathbf{i}-\mathbf{k} & -1-\mathbf{j}+\mathbf{k} \\ -1+5\mathbf{i}-\mathbf{j}+\mathbf{k} & 3+3\mathbf{i}-\mathbf{j}-\mathbf{k} & -1+\mathbf{i}+2\mathbf{k} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & \mathbf{i}+\mathbf{k} & 1+2\mathbf{i}-\mathbf{j} \\ \mathbf{i} & -1-\mathbf{j} & -2+\mathbf{i}-\mathbf{k} \\ 1+\mathbf{i} & -1+\mathbf{i}+2\mathbf{k} & -1+3\mathbf{i}-2\mathbf{j} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -1+4\mathbf{i}-\mathbf{j}-\mathbf{k} & -4+2\mathbf{i}-5\mathbf{j}+6\mathbf{k} & 3-2\mathbf{i}+6\mathbf{k} \\ 1-5\mathbf{i}-6\mathbf{j}+\mathbf{k} & 5+\mathbf{i}-2\mathbf{j}+\mathbf{k} & 3-2\mathbf{i}+\mathbf{k} \\ -6-3\mathbf{i}+2\mathbf{j}+3\mathbf{k} & -2-8\mathbf{i}+3\mathbf{j}+11\mathbf{k} & 5\mathbf{j} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 2-3\mathbf{i} & 8-3\mathbf{j}+4\mathbf{k} & -1+\mathbf{i}-5\mathbf{j}-8\mathbf{k} \\ 1-2\mathbf{j} & 1-9\mathbf{i}-4\mathbf{j}-2\mathbf{k} & -6+2\mathbf{i}-\mathbf{j}-5\mathbf{k} \\ \mathbf{j}-2\mathbf{k} & 6-8\mathbf{i}-5\mathbf{j}+2\mathbf{k} & -7-\mathbf{i}-5\mathbf{j}-4\mathbf{k} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 3+3\mathbf{j}-\mathbf{k} & -3+\mathbf{i}+6\mathbf{j}-2\mathbf{k} & 1-4\mathbf{j}-5\mathbf{k} \\ 1+\mathbf{i}+\mathbf{j}+2\mathbf{k} & 6-4\mathbf{i}+4\mathbf{j}+3\mathbf{k} & -6+13\mathbf{j}-4\mathbf{k} \\ 3+4\mathbf{i}-\mathbf{j}+6\mathbf{k} & 3-3\mathbf{i}+5\mathbf{j}-4\mathbf{k} & 2-\mathbf{i}+5\mathbf{j}-7\mathbf{k} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -11-5\mathbf{i}-6\mathbf{j}+\mathbf{k} & -1+8\mathbf{i}-2\mathbf{j}+7\mathbf{k} & -10+\mathbf{i}-3\mathbf{j}+6\mathbf{k} \\ 5-11\mathbf{i}-3\mathbf{j}-5\mathbf{k} & -6-2\mathbf{i}-5\mathbf{j}-3\mathbf{k} & -2-12\mathbf{i}-3\mathbf{j}+3\mathbf{k} \\ -6-16\mathbf{i}-5\mathbf{j}-4\mathbf{k} & -11+6\mathbf{i}-7\mathbf{j} & -12-7\mathbf{i}-2\mathbf{j}+\mathbf{k} \end{pmatrix}.$$

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9). Check that

$$\begin{aligned}
r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} &= r(A_i) + r(B_i) = \begin{cases} 6, & \text{if } i = 1 \\ 4, & \text{if } i = 2, 3 \\ 3, & \text{if } i = 4 \end{cases} \\
r \begin{pmatrix} A_{k+1}C_k + C_{k+1}B_k & A_{k+1}A_k \\ B_{k+1}B_k & 0 \end{pmatrix} &= r(A_{k+1}A_k) + r(B_{k+1}B_k) = \begin{cases} 4, & \text{if } k = 1, 2 \\ 3, & \text{if } k = 3 \end{cases} \\
r \begin{pmatrix} A_{j+2}A_{j+1}C_j + A_{j+2}C_{j+1}B_j + C_{j+2}B_{j+1}B_j & A_{j+2}A_{j+1}A_j \\ B_{j+2}B_{j+1}B_j & 0 \end{pmatrix} \\
&= r(A_{j+2}A_{j+1}A_j) + r(B_{j+2}B_{j+1}B_j) = \begin{cases} 4, & \text{if } j = 1 \\ 3, & \text{if } j = 2 \end{cases} \\
r \begin{pmatrix} A_4A_3A_2C_1 + A_4A_3C_2A_1 + A_4C_3A_2A_1 + C_4A_3A_2A_1 & A_4A_3A_2A_1 \\ B_4B_3B_2B_1 & 0 \end{pmatrix} \\
&= r(A_4A_3A_2A_1) + r(B_4B_3B_2B_1) = 3.
\end{aligned}$$

All the rank equalities in (6.1)-(6.4) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) is consistent. Note that

$$\begin{aligned}
X_1 &= \begin{pmatrix} 2\mathbf{i} + \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & 2 + \mathbf{j} \\ -2\mathbf{i} + \mathbf{k} & 1 + \mathbf{j} + \mathbf{k} & -2 + \mathbf{j} \\ 2\mathbf{k} & 2\mathbf{j} + 2\mathbf{k} & 2\mathbf{j} \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 & -1 + \mathbf{j} & \mathbf{i} + \mathbf{k} \\ 2 & -2 - \mathbf{j} & 2\mathbf{i} - \mathbf{k} \\ -1 & 1 + 2\mathbf{j} & -\mathbf{i} + 2\mathbf{k} \end{pmatrix}, \\
X_3 &= \begin{pmatrix} \mathbf{i} + \mathbf{j} & 1 + 2\mathbf{i} + \mathbf{k} & 2\mathbf{k} \\ 1 & \mathbf{k} & 1 \\ \mathbf{i} & 0 & 1 + \mathbf{k} \end{pmatrix} \quad X_4 = \begin{pmatrix} -1 + \mathbf{i} + \mathbf{k} & 1 + \mathbf{k} & \mathbf{i} + \mathbf{k} \\ -1 - \mathbf{i} + \mathbf{k} & \mathbf{i} + \mathbf{k} & -1 + \mathbf{k} \\ -2 + 2\mathbf{k} & 1 + \mathbf{i} + 2\mathbf{k} & -1 + \mathbf{i} + 2\mathbf{k} \end{pmatrix},
\end{aligned}$$

and

$$X_5 = \begin{pmatrix} 1 & -1 + \mathbf{j} & \mathbf{i} + \mathbf{k} \\ 2 & -2 + 2\mathbf{j} & 2\mathbf{i} + 2\mathbf{k} \\ 3 & -3 - \mathbf{j} & 3\mathbf{i} - \mathbf{k} \end{pmatrix}$$

satisfy the system (1.9).

7. Some solvability conditions and the general solution to system (1.10)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10). For simplicity, put

$$A_{11} = R_{B_2}B_1, \quad B_{11} = R_{A_1}A_2, \quad C_{11} = B_1L_{A_{11}}, \quad D_{11} = R_{A_1}(R_{A_2}C_2B_2^\dagger B_1 - C_1)L_{A_{11}},$$

$$A_{22} = R_{(A_4L_{A_3})}A_4, \quad B_{22} = B_3L_{B_4}, \quad C_{22} = R_{(A_4L_{A_3})}(C_4 - A_4A_3^\dagger C_3)L_{B_4},$$

$$A_{33} = (L_{A_2}, -A_3 L_{A_{22}}), \quad B_{33} = \begin{pmatrix} R_{C_{11}} B_2 \\ -R_{B_3} \end{pmatrix},$$

$$E_1 = -R_{A_3} C_3 B_3^\dagger + A_3 A_{22}^\dagger C_{22} B_{22}^\dagger - A_2^\dagger C_2 - B_{11}^\dagger D_{11} C_{11}^\dagger B_2,$$

$$A = R_{A_{33}} L_{B_{11}}, \quad B = B_2 L_{B_{33}}, \quad C = -R_{A_{33}} A_3, \quad D = R_{B_{22}} L_{B_{33}},$$

$$E = R_{A_{33}} E_1 L_{B_{33}}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M.$$

Theorem 7.1. *Let A_i, B_i , and $C_i (i = 1, 2, 3, 4)$ be given. Then the following statements are equivalent:*

(1) *The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) is consistent.*

(2)

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \quad (7.1)$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2), \quad (7.2)$$

$$r \begin{pmatrix} C_3 & A_3 \\ C_4 & A_4 \\ B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_3 \\ A_4 \end{pmatrix} + r \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}, \quad (7.3)$$

$$r \begin{pmatrix} C_1 & A_2 C_3 + C_2 B_3 & A_1 & A_2 A_3 \\ B_1 & B_2 B_3 & 0 & 0 \end{pmatrix} = r(A_1, A_2 A_3) + r(B_1, B_2 B_3), \quad (7.4)$$

$$r \begin{pmatrix} A_2 C_3 + C_2 B_3 & A_2 A_3 \\ C_4 & A_4 \\ B_2 B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 A_3 \\ A_4 \end{pmatrix} + r \begin{pmatrix} B_2 B_3 \\ B_4 \end{pmatrix}, \quad (7.5)$$

$$r \begin{pmatrix} A_2 C_3 + C_2 B_3 & A_2 A_3 \\ B_2 B_3 & 0 \end{pmatrix} = r(A_2 A_3) + r(B_2 B_3), \quad (7.6)$$

$$r \begin{pmatrix} C_1 & A_2 C_3 + C_2 B_3 & A_1 & A_2 A_3 \\ 0 & C_4 & 0 & A_4 \\ B_1 & B_2 B_3 & 0 & 0 \\ 0 & B_4 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 A_3 \\ 0 & A_4 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 B_3 \\ 0 & B_4 \end{pmatrix}. \quad (7.7)$$

(3)

$$R_{A_2} C_2 L_{B_2} = 0, \quad D_{11} L_{C_{11}} = 0, \quad R_{B_{11}} D_{11} = 0,$$

$$R_{A_3}C_3L_{B_3} = 0, \quad R_{A_{22}}C_{22} = 0, \quad C_{22}L_{B_{22}} = 0,$$

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) can be expressed as

$$X_1 = A_1^\dagger(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6,$$

$$X_2 = -R_{A_2}C_2B_2^\dagger + A_2U_1 + V_1R_{B_2}, \quad X_4 = A_3^\dagger C_3 + V_2B_3 + L_{A_3}U_2,$$

$$X_5 = -R_{(A_4L_{A_3})}(C_4 - A_4A_3^\dagger C_3 - A_4V_2B_3)B_4^\dagger + A_4L_{A_3}T_1 + T_3R_{B_4},$$

$$X_3 = A_2^\dagger C_2 + U_1B_2 + L_{A_2}W_1, \quad \text{or } X_3 = -R_{A_3}C_3B_3^\dagger + A_3V_2 + T_6R_{B_3},$$

where

$$U_1 = B_{11}^\dagger D_{11}C_{11}^\dagger + L_{B_{11}}W_2 + W_3R_{C_{11}},$$

$$V_1 = -R_{A_1}(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1)A_{11}^\dagger + A_1W_4 + W_5R_{A_{11}},$$

$$V_2 = A_{22}^\dagger C_{22}B_{22}^\dagger + L_{A_{22}}T_4 + T_5R_{B_{22}},$$

$$U_2 = (A_4L_{A_3})^\dagger(C_4 - A_4A_3^\dagger C_3 - A_4V_2B_3) + T_1B_4 + L_{(A_4L_{A_3})}T_2,$$

$$W_1 = (I_{p_1}, 0)[A_{33}^\dagger(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}}) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$T_4 = (0, I_{p_2})[A_{33}^\dagger(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}}) - A_{33}^\dagger Z_7B_{33} + L_{A_{33}}Z_6],$$

$$W_3 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}})B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix},$$

$$T_6 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}})B_{33}^\dagger + A_{33}A_{33}^\dagger Z_7 + Z_8R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix},$$

$$W_2 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1R_NDB^\dagger + L_AZ_2 + Z_3R_B,$$

$$T_5 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_SZ_4 + L_MZ_1R_N + Z_5R_D,$$

the remaining W_j, T_j, Z_j are arbitrary matrices over \mathbb{H} , p_1 and p_2 are the column numbers of A_2 and A_4 , respectively, p_3 and p_4 are the row numbers of B_1 and B_3 , respectively.

Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) into two parts

$$\begin{cases} A_1 X_1 - X_2 B_1 = C_1, \\ A_2 X_3 - X_2 B_2 = C_2, \end{cases} \quad (7.8)$$

and

$$\begin{cases} A_3 X_4 - X_3 B_3 = C_3, \\ A_4 X_4 - X_5 B_4 = C_4. \end{cases} \quad (7.9)$$

Applying the main idea of Theorem 3.1, Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5, we can prove Theorem 7.1. \square

Now we give an example to illustrate Theorem 7.1.

Example 5. *Given the quaternion matrices:*

$$\begin{aligned} A_1 &= \begin{pmatrix} \mathbf{i} + \mathbf{j} & -\mathbf{j} & \mathbf{i} + \mathbf{k} \\ \mathbf{k} & 1 + \mathbf{k} & 0 \\ 1 & 0 & 1 + \mathbf{j} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 + \mathbf{j} + \mathbf{k} & -1 - \mathbf{k} & \mathbf{i} + \mathbf{j} \\ 2\mathbf{k} & 1 & 1 + \mathbf{i} + \mathbf{j} \\ 2 & 2 + \mathbf{i} + \mathbf{j} & \mathbf{k} \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 1 & 1 + \mathbf{i} + \mathbf{j} & 2 + 2\mathbf{i} + \mathbf{k} \\ 1 - 2\mathbf{i} + \mathbf{k} & \mathbf{j} & 1 \\ \mathbf{i} & 1 & \mathbf{j} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -1 + \mathbf{j} & \mathbf{i} + \mathbf{k} \\ \mathbf{i} & -\mathbf{i} - \mathbf{j} & -1 - \mathbf{k} \\ 1 + \mathbf{i} & -1 - \mathbf{i} & -1 + \mathbf{i} \end{pmatrix}, \\ A_3 &= \begin{pmatrix} \mathbf{j} & \mathbf{i} - \mathbf{j} & 1 + \mathbf{k} \\ 1 + \mathbf{k} & 0 & \mathbf{i} + \mathbf{j} \\ 1 + \mathbf{j} + \mathbf{k} & \mathbf{i} - \mathbf{j} & 1 + \mathbf{i} + \mathbf{j} + \mathbf{k} \end{pmatrix}, \quad B_3 = \begin{pmatrix} \mathbf{j} + 2\mathbf{k} & 1 + \mathbf{j} - \mathbf{k} & \mathbf{i} + \mathbf{j} \\ -\mathbf{j} - 2\mathbf{k} & -1 - \mathbf{j} + \mathbf{k} & -\mathbf{i} - \mathbf{j} \\ 2\mathbf{j} + 4\mathbf{k} & 2\mathbf{j} - 2\mathbf{k} & 2\mathbf{j} \end{pmatrix}, \\ A_4 &= \begin{pmatrix} -\mathbf{k} & \mathbf{i} + \mathbf{j} + \mathbf{k} & 2\mathbf{i} - 2\mathbf{k} \\ 1 + \mathbf{k} & 1 - \mathbf{j} - \mathbf{k} & 1 + 2\mathbf{k} \\ 1 & 1 + \mathbf{i} & 1 + 2\mathbf{i} \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 - \mathbf{j} & \mathbf{i} - \mathbf{k} & -\mathbf{i} - \mathbf{k} \\ \mathbf{i} + \mathbf{j} & -1 + \mathbf{k} & 1 + \mathbf{k} \\ 1 + \mathbf{i} + 2\mathbf{j} & -1 + \mathbf{i} + 2\mathbf{k} & 1 - \mathbf{i} + 2\mathbf{k} \end{pmatrix}, \\ C_1 &= \begin{pmatrix} -3 - \mathbf{j} - 4\mathbf{k} & -1 + 2\mathbf{i} + 3\mathbf{k} & 1 - \mathbf{j} \\ 2 - \mathbf{i} + \mathbf{j} - 3\mathbf{k} & -1 + \mathbf{i} - \mathbf{k} & 2 - \mathbf{i} + \mathbf{k} \\ 4 + \mathbf{i} - 3\mathbf{k} & 1 - 2\mathbf{i} + \mathbf{j} & -2 - 2\mathbf{i} - \mathbf{j} - \mathbf{k} \end{pmatrix}, \\ C_2 &= \begin{pmatrix} -2 - 2\mathbf{i} - 8\mathbf{j} + 5\mathbf{k} & 11\mathbf{j} + 7\mathbf{k} & 2 + \mathbf{i} - 8\mathbf{j} + 5\mathbf{k} \\ -1 - 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} & -1 + 2\mathbf{i} + 9\mathbf{j} - \mathbf{k} & 4 + \mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \\ 1 - 2\mathbf{j} + \mathbf{k} & -3 + \mathbf{i} + 5\mathbf{j} - \mathbf{k} & 2 + 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} \end{pmatrix}, \\ C_3 &= \begin{pmatrix} -4 + 9\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} & -8 - 3\mathbf{i} + \mathbf{k} & -5 + \mathbf{j} - 6\mathbf{k} \\ 8 + 5\mathbf{i} - \mathbf{j} + 6\mathbf{k} & -2 + 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} & -2 + 6\mathbf{i} + 2\mathbf{k} \\ 4 + 14\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} & -10 + \mathbf{i} + 6\mathbf{j} + \mathbf{k} & -7 + 6\mathbf{i} + \mathbf{j} - 4\mathbf{k} \end{pmatrix}, \end{aligned}$$

$$C_4 = \begin{pmatrix} -3 + 2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k} & 2 + 4\mathbf{i} + \mathbf{j} - 3\mathbf{k} & -4\mathbf{i} - \mathbf{j} - \mathbf{k} \\ -3 - 4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k} & 7 - 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} & -10 + 8\mathbf{i} + 3\mathbf{j} - 9\mathbf{k} \\ -4 + \mathbf{i} - 3\mathbf{j} + 3\mathbf{k} & 4 + 3\mathbf{i} - 3\mathbf{j} & -7 + 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \end{pmatrix}.$$

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10). Check that

$$r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 6, & \text{if } i = 1 \\ 5, & \text{if } i = 2 \\ 4, & \text{if } i = 3, 4 \end{cases}$$

$$r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2) = 6,$$

$$r \begin{pmatrix} C_3 & A_3 \\ C_4 & A_4 \\ B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_3 \\ A_4 \end{pmatrix} + r \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} = 6,$$

$$r \begin{pmatrix} C_1 & A_2C_3 + C_2B_3 & A_1 & A_2A_3 \\ B_1 & B_2B_3 & 0 & 0 \end{pmatrix} = r(A_1, A_2A_3) + r(B_1, B_2B_3) = 6,$$

$$r \begin{pmatrix} A_2C_3 + C_2B_3 & A_2A_3 \\ C_4 & A_4 \\ B_2B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r \begin{pmatrix} A_2A_3 \\ A_4 \end{pmatrix} + r \begin{pmatrix} B_2B_3 \\ B_4 \end{pmatrix} = 6,$$

$$r \begin{pmatrix} A_2C_3 + C_2B_3 & A_2A_3 \\ B_2B_3 & 0 \end{pmatrix} = r(A_2A_3) + r(B_2B_3) = 10,$$

$$r \begin{pmatrix} C_1 & A_2C_3 + C_2B_3 & A_1 & A_2A_3 \\ 0 & C_4 & 0 & A_4 \\ B_1 & B_2B_3 & 0 & 0 \\ 0 & B_4 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2A_3 \\ 0 & A_4 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2B_3 \\ 0 & B_4 \end{pmatrix} = 4.$$

All the rank equalities in (7.1)-(7.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) is consistent. Note that

$$X_1 = \begin{pmatrix} 1 + \mathbf{i} + \mathbf{j} & 1 - 2\mathbf{i} + 2\mathbf{j} & \mathbf{k} \\ 1 & \mathbf{i} + \mathbf{j} & 2 \\ \mathbf{i} + \mathbf{k} & 1 + 2\mathbf{k} & -\mathbf{k} \end{pmatrix} \quad X_2 = \begin{pmatrix} \mathbf{j} & 1 - \mathbf{j} & \mathbf{i} + \mathbf{k} \\ \mathbf{i} + \mathbf{j} & 1 - \mathbf{j} & \mathbf{k} \\ -1 & 2 + \mathbf{k} & \mathbf{i} + \mathbf{j} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -1 - \mathbf{j} + \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & \mathbf{i} - \mathbf{j} \\ -1 + \mathbf{j} + \mathbf{k} & 1 + \mathbf{j} - \mathbf{k} & -\mathbf{i} + \mathbf{k} \\ 2\mathbf{k} & 2\mathbf{j} & -\mathbf{j} + \mathbf{k} \end{pmatrix} \quad X_4 = \begin{pmatrix} 1 & 1 + \mathbf{j} & \mathbf{i} + \mathbf{k} \\ \mathbf{i} & \mathbf{i} - \mathbf{j} & -1 - \mathbf{k} \\ 1 + \mathbf{i} & 1 + \mathbf{i} & -1 + \mathbf{i} \end{pmatrix},$$

and

$$X_5 = \begin{pmatrix} 1 + \mathbf{j} & 1 + \mathbf{j} & \mathbf{i} + \mathbf{k} \\ 1 + \mathbf{i} + \mathbf{k} & 2 - \mathbf{i} + \mathbf{k} & 3 \\ 1 + 3\mathbf{i} + \mathbf{j} & \mathbf{k} & 1 \end{pmatrix}$$

satisfy the system (1.10).

8. CONCLUSION

We have provided some necessary and sufficient conditions for the existence and the general solutions to the systems of four coupled one sided Sylvester-type real quaternion matrix equations (1.6)-(1.10), respectively. Moreover, we have presented some numerical examples. It is worthy to say that the main results of this paper can be generalized to an arbitrary division ring with an involutive antiautomorphism.

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